INTERNAL WAVE FIELD GENERATED BY A SOURCE AT REST
IN A MOVING STRATIFIED FLUID

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We consider the generation of internal waves by a source in a uniform stream of stratified fluid. Asymptotic methods are usually used in studying the internal wave field [1, 2]. An exact solution of this problem was found in [3] in the form of quadratures of special functions and numerical results were given for the case of constant Väisälä–Brunt frequency N(z). The case of a two-layer fluid was considered in [4]. In the present paper we give new, simpler quadrature formulas for the field, present numerical results based on these formulas for arbitrary N(z), and discuss the local features of the wave field near the source.

The field of internal waves in a layer $0 < z < H$ produced by a source at rest in a stream of stratified fluid is described in the Boussineq approximation by the equation

$$\frac{\partial^2}{\partial t^2}(\eta_{xx}+\eta_{yy}+\eta_{zz})+N^2(z)(\eta_{xx}+\eta_{yy}) = Q\theta(t)\,\delta_t'(x+vt)\,\delta(y)\,\delta'(z-z_0), \tag{1}$$

where $\eta(x, y, z, t)$ is the elevation and is related to the vertical component of the velocity $w(x, y, z, t)$ by the relation $w = \partial\eta/\partial t$; v is the velocity of the stream; z_0 is the depth of the source; $\theta(t) = 0$ for $t \le 0$ and $\theta(t) = 1$ for $t > 0$; Q is the strength of the source.

The boundary conditions are written in the approximation of rigidly fixed top and bottom boundaries of the layer

$$\eta = 0,\ z = 0,\ H. \tag{2}$$

It was shown in [5] that the solution of (1) and (2) can be represented as a sum of modes, each of which has a maximum group velocity c_n ($c_1 > c_2 > \ldots$). We consider the most common case when $v > c_n$, $n = 1, 2, \ldots$ Then the solution of problem (1), (2) takes the form [5]

$$\eta = \sum_n \eta_n = \frac{Q}{4\pi^2}\sum_n \int_{-\infty}^{\infty} d\nu \times \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} d\mu \frac{i\mu v\omega_n^2(k)\,\varphi_n(z,k)\exp(-i\mu(x+vt)-i\nu y)}{k^2\left(\omega_n^2(k)-\mu^2v^2\right)}\frac{\partial\varphi_n(z_0,k)}{\partial z_0}. \tag{3}$$

Here $\varphi_n(z, k)$ and $\omega_n(k)$ are the eigenfunctions and eigenvalues of the problem

$$\frac{\partial^2\varphi_n}{\partial z^2}+k^2\left[\frac{N^2(z)}{\omega_n^2(k)}-1\right]\varphi_n = 0,\ \varphi_n = 0\ \ (z = 0,\ H). \tag{4}$$

When $v < c_1$ the solution is more complicated and this case requires a separate treatment.

The inner integral in (3) is evaluated by closing the contour of integration with respect to μ in the upper halfplane for $\xi < 0$ ($\xi = x + vt$) and in the lower halfplane for $\xi > 0$. We consider the locations of the singular points (poles and branch points) of the integrand in (3) for real ν and complex μ.

It can be shown that poles of this expression [the roots of the equation $\mu^2v^2 = \omega_n^2(\sqrt{\nu^2+\mu^2})$] exist only for real μ^2, i.e., μ must either be real [$\mu = \pm\mu_n(\nu)$] or purely

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imaginary $[\mu = \pm i\lambda_n(\nu)]$. Indeed, it follows from (3) that the roots of the equation $\mu^2 v^2 = \omega_n{}^2(\sqrt{\nu^2 + \mu^2})$ are the eigenvalues of the problem

$$\frac{\partial^2 \psi_n(z,\nu)}{\partial z^2} + \left\{\left[\frac{N^2(z)}{v^2} - \nu^2\right] + \left[\frac{\nu^2 N^2(z)}{v^2 \mu_n^2(\nu)} - \mu_n^2(\nu)\right]\right\}\psi_n(z,\nu) = 0,$$
$$\psi_n = 0 \quad (z = 0, H), \tag{5}$$

which is not a classical Sturm–Liouville problem, since the spectral parameter $\mu_n(\nu)$ does not enter in the standard way. Multiplying (5) by the complex-conjugate function $\overline{\psi_n(z,\nu)}$ and integrating with respect to z, we obtain

$$\int_0^H \left\{-\left|\frac{\partial \psi_n(z,\nu)}{\partial z}\right|^2 + \left[\frac{N^2(z)}{v^2} - \nu^2\right]|\psi_n(z,\nu)|^2 + \right.$$
$$\left. + \left[\frac{\nu^2 N^2(z)}{v^2 \mu_n^2(\nu)} - \mu_n^2(\nu)\right]|\psi_n(z,\nu)|^2\right\} dz = 0.$$

When $\operatorname{Im}\mu_n{}^2(\nu) \neq 0$, the imaginary part of the integrand has the opposite sign of $\operatorname{Im}\mu_n{}^2(\nu)$ and the integral does not vanish. This contradiction proves that the spectral problem (5) only exists for real $\mu_n{}^2(z)$. An analogous treatment can be given for $\mu = \pm i\lambda_n(\nu)$, where $\lambda_n(\nu)$ are the eigenvalues of the problem

$$\frac{\partial^2 f_n(z,\nu)}{\partial z^2} + \left\{\left[\frac{N^2(z)}{v^2} - \nu^2\right] + \left[\lambda_n^2(\nu) - \frac{\nu^2 N^2(z)}{v^2 \lambda_n^2(\nu)}\right]\right\} f_n(z,\nu) = 0,$$
$$f_n = 0 \quad (z = 0, H). \tag{6}$$

If $v > c_1$, then eigenvalues $\lambda_n(\nu)$ and eigenfunctions $f_n(z, \nu)$ exist for the spectral problem (5) for any ν.

The function $\omega_n{}^2(\sqrt{\nu^2 + \mu^2})$ can also have branch points for complex values $\mu = \mu_*(\nu)$, for which $\partial\omega/\partial k$ diverges at $k = k_* = \sqrt{\nu^2 + \mu_*{}^2(\nu)}$. Differentiating (4) with respect to the parameter k, it is not difficult to show that for this to occur we must have $\int_0^H N^2(z)\varphi_n^2(z, k)dz = 0$.

Hence, in the integration with respect to μ in (3), the contribution of the integrals along the cuts must be taken into account, in addition to the contributions of the poles at the points $\mu = \pm\mu_n(\nu)$ and $\mu = \pm i\lambda_n(\nu)$. However, the integrals along the cuts cancel one another out in the summation over n in (3).

Indeed, let $\mu = \mu_*$ be a branch point of the function $\omega_n(\sqrt{\nu^2 + \mu^2})$. Since each of the branches of the function $\omega_n(\sqrt{\nu^2 + \mu^2})$ is obviously an eigenvalue of the spectral problem (4), there are several eigenvalues which join into a single value in the limit $\mu \to \mu_*$ and transform into one another when going around the branch point $\mu = \mu_*$. In the simplest case there will be two such eigenvalues: ω_n and ω_m, which transform into one another when going around the branch point. When going around the branch point in the sum (3), the term η_n transforms into η_m, η_m transforms into η_n, and the sum $\eta_n + \eta_m$ obviously transforms into itself. In other words, for the sum $\eta_n + \eta_m$ the point $\mu = \mu_*$ is a removable singular point and, therefore, the integral along the cut vanishes. Similarly, one can show that the integrals along the cuts can be neglected in the case of higher-order branch points. Therefore, the integrals with respect to μ in each of the terms of (3) can be evaluated taking into account only the poles of the integrand. Furthermore, we will consider the case of a single mode. Closing the contour of integration with respect to μ in the upper halfplane for $\xi < 0$ and in the lower halfplane for $\xi > 0$, it can be shown that $\eta_n = I_+ + I_- + I_0$ $(\xi > 0)$ or $\eta_n = -I_0$ $(\xi < 0)$. Here

$$I_\pm = -\frac{Q}{2\pi i}\int_{-\infty}^{\infty} \exp(\pm i\mu_n(\nu)\xi - i\nu y)\, b_n(\nu)\,\psi_n(z,\nu)\frac{\partial\psi_n(z_0,\nu)}{\partial z_0}d\nu; \tag{7}$$

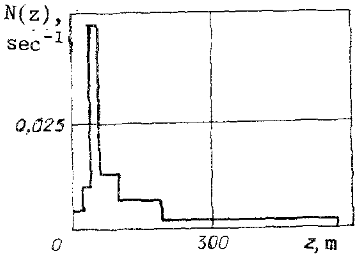

Fig. 1

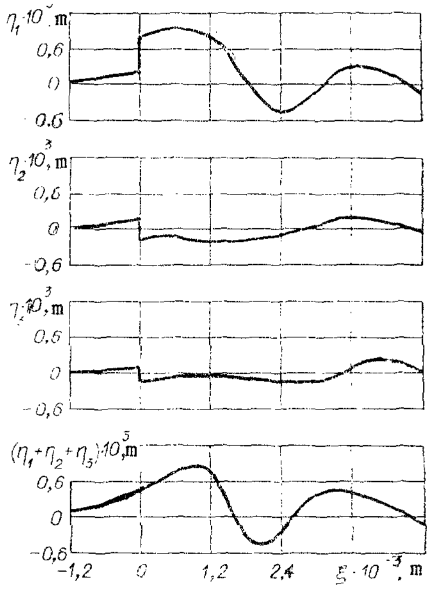

Fig. 2

$$I_0 = -\frac{Q}{2\pi i}\int\limits_{-\infty}^{\infty} \exp\left(-\lambda_n(\nu)|\xi| + i\nu y\right) d_n(\nu) f_n(z, \nu)\,\frac{\partial f_n(z_0, \nu)}{\partial z_0}\,d\nu,$$

$$b_n(\nu) = \frac{i\mu_n^2(\nu)\,v}{2\left(\mu_n^2(\nu)+\nu^2\right)}\left(\frac{\mu_n(\nu)\,\mu_n'(\nu)}{\nu}+1\right), \quad d_n(\nu) = \frac{\lambda_n^2(\nu)\,v}{2i\left(\lambda_n^2(\nu)-\nu^2\right)}\left(\frac{\lambda_n(\nu)\,\lambda_n'(\nu)}{\nu}-1\right). \tag{8}$$

The properties of the functions $\mu_n(\nu)$ have been described in [5]. We consider the function $\lambda_n(\nu)$. It is not difficult to show, using the perturbation method, that for small ν, $\lambda_n(\nu)$ can be expanded in a series of even powers of ν: $\lambda_n(\nu) = \alpha_n + \beta_n\nu^2 + \ldots$, where α_n are determined from the Sturm–Liouville problem

$$\frac{d^2 p_n}{dz^2} + \left[\frac{N^2(z)}{v^2} + \alpha_n^2\right] p_n = 0, \quad p_n = 0 \quad (z = 0, H),$$

in which $p_n(z) = f_n(z, 0)$ and

$$\beta_n = \frac{1}{2\alpha_n}\left(1 + \frac{l_n^2}{v^2\alpha_n^2}\right), \quad l_n^2 = \frac{\int\limits_0^H N^2(z)\,p_n^2(z)\,dz}{\int\limits_0^H p_n^2(z)\,dz}.$$

Similarly, it can be shown that in the limit $\nu \to \infty$ the function $\lambda_n(\nu)$ has the expansion

$$\lambda_n(\nu) = \nu + \gamma_n/\nu + \delta_n/\nu^3 + \ldots,$$

$$\gamma_n = \frac{\pi^2 n^2}{2H^2}, \quad \delta_n = -\frac{\pi^2 n^2}{H^3}\left(\frac{\pi^2 n^2}{8H} + \frac{1}{v^2}\int\limits_0^H N^2(z)\sin^2\frac{n\pi}{H}z\,dz\right).$$

We used the N(z) distribution shown in Fig. 1 in the numerical calculations. The results of the calculations using Eqs. (7) and (8) with v = 2 m/sec, y = 300 m, Q = 1 m^3/sec, z = 200 m, z_0 = 22 m are shown in Fig. 2. Calculations were performed for the first three modes. We see that the separate terms η_n (n = 1, 2, 3) are all discontinuous at $\xi = 0$, but their sum is continuous. For large y the main contribution to the field results from the term I_+; the other terms, including the integral along the cut, are negligibly small and the functions become practically continuous. Hence, in the numerical calculation of the near field, the continuity of the sum can serve as a criterion for determining the number of modes contributing to the total field.

We note that in the calculation of the vertical velocity field w, the individual modes are always continuous, since the integrals along the cuts contribute with the same sign when closing the contour of integration upward ($\xi < 0$) and downward ($\xi > 0$). In this case the branch points appear as pairs and the integrand is odd in μ. The above criterion cannot be used in this case.

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TURBULENT FLOATING JET IN A STRATIFIED ATMOSPHERE

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The reliability of predictions of the ecological consequences of a number of natural and anthropogenic catastrophes (volcano eruptions, large fires, atomic electric station emergencies, nuclear explosions) depends to a significant extent on the accuracy of predicting the initial spatial pattern of atmospheric pollution above individual heat and impurity sources [1]. The maximal altitude of impurity ejection and its concentration distribution in space at a time near to termination of average vertical freely convective movements of clouds or jets of heated products are understood to be the initial pollution.

Depending on the relationship between the time of heat (impurity) source action t_S and the characteristic time of thermal relaxation of the atmosphere $t_N \approx 2\pi N^{-1}$ (N is the Väisälä-Brunt frequency), two limiting spatial configurations of freely convective motions [2] can be separated out. If $t_S \ll t_N$ (in the limit for instantaneous energy liberation) then a floating cloud, a thermal, severed from the earth, is formed rapidly in the atmosphere. A convective column of an ascending jet movement of the products is formed above the focus for the reverse relationship between the times (in the limit for a permanently acting source). For the standard state of the atmosphere ($N = 0.0106\ \text{sec}^{-1}$ in the tropospheric layer) $t_N \approx$ 10 min. During this time the cloud or jet reaches its maximal point of ascent and starts to be deformed in mainly a horizontal direction. The thermal will here perform damping vibrational vertical motions around the thermal equilibrium level, while the convective column (from a fire focus, say) will form a slowly expanding configuration of a quasistationary jet flow at the altitude of hanging.

The transport of impurities in the atmosphere by powerful thermals is investigated in sufficient detail by both analytic [2-5] and numerical [6, 7] methods and the predictions of theory are mainly in good agreement with experimental results. Results of studying the second limit case of freely convective motions, the two-dimensional axisymmetric turbulent floating jet, are elucidated below.

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